TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 8, Pages 3229–3246
S 0002-9947(01)02555-7
Article electronically published on April 9, 2001

LINK COMPLEMENTS AND THE BIANCHI MODULAR GROUPS

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Abstract. We determine the values of m for which the Bianchi modular group $\mathrm{PSL}_2(\mathcal{O}_m)$ contains a link group.

1. Introduction and statement of results

Let $\mathcal{O}_m$ be the integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$. The Bianchi modular group, $\mathrm{PSL}_2(\mathcal{O}_m)$, is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ the orientation preserving isometries of hyperbolic 3-space, $\mathbb{H}^3$.

In this paper we determine those values of m for which $\mathrm{PSL}_2(\mathcal{O}_m)$ contains a link group, that is, a torsion-free subgroup of finite index, Γ, such that $\mathbb{H}^3/\Gamma$ is homeomorphic to a link complement in S^3.

Results on the cohomology of $\mathrm{PSL}_2(\mathcal{O}_m)$ (see [V]) limit the values of m for which this is possible to the following list, $\mathcal{L}$:

$$\mathcal{L} = \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}.$$

Over the last 20 years, numerous link groups have been found in $\mathrm{PSL}_2(\mathcal{O}_m)$ for the seven cases $m = 1, 2, 3, 7, 11, 15, 23$ (see [B], [H1], [R1], [T], [Wi]). Furthermore, many of the corresponding links (figure eight knot, Whitehead link, Borromean rings,...) have been central to the study of 3-manifolds.

We prove that the seven remaining Bianchi groups from the above list, $\mathcal{L}$, also contain link groups; hence:

Theorem. *The Bianchi group* $\mathrm{PSL}_2(\mathcal{O}_m)$ *contains a link group if and only if* $m \in \mathcal{L}$.

2. Preliminaries and outline of proof

Henceforth, let Ω_m denote the orbifold $\mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O}_m)$. We prove our result by:

i) Embedding Ω_m in S^3 as the complement of a link of circles. The orbifolds Ω_m for $m = 5, 6, 15, 19, 23, 31, 39, 47, 71$ are drawn in Figure 1.

ii) Using the embeddings in (i) to prove that Ω_m admits a finite sheeted orbifold covering $\widetilde{\Omega}_m \to \Omega_m$ such that $\widetilde{\Omega}_m$ is a nonsingular hyperbolic link complement.

Thus $\widetilde{\Omega}_m \cong \mathbb{H}^3/\Gamma$, where Γ is a torsion-free finite index subgroup of $\mathrm{PSL}_2(\mathcal{O}_m)$ and hence Γ is a link group (see [T], Chapter 13). The existence of $\widetilde{\Omega}_m$ follows from an analysis of the singular locus of Ω_m.

In Section 3 we construct the link complements, $\widetilde{\Omega}_m$. We also apply our methods to construct link complements corresponding to subgroups of $\mathrm{PSL}_2(\mathcal{O}_{15})$ and

Received by the editors June 3, 1998 and, in revised form, March 8, 1999.
2000 *Mathematics Subject Classification.* Primary 57M25; Secondary 11F06.
Key words and phrases. Link complements, Bianchi modular groups.

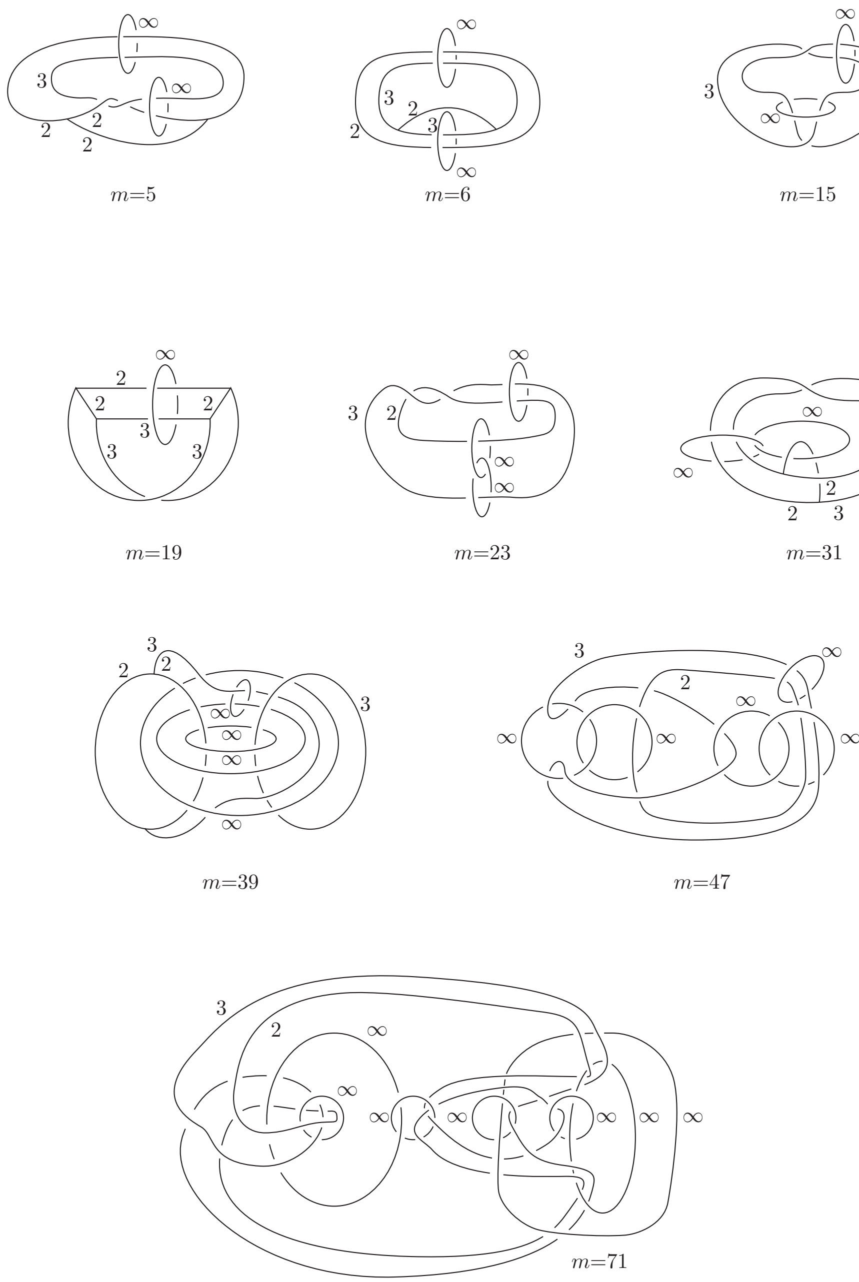


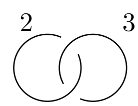
FIGURE 1. The orbifolds Ω_m

$\mathrm{PSL}_2(\mathcal{O}_{23})$ giving further examples in these two cases (see [B] for previous examples). Section 4 is devoted to a discussion of the orbifolds Ω_m and their embeddings in S^3. In Section 5 we give the volumes of $\widetilde{\Omega}_m$ calculated using SnapPea. Comparing these volumes with expected theoretical values provides a good check of our results.

3. Link complements: The covers $\widetilde{\Omega}_m$

Consider the orbifolds Ω_m drawn in Figure 1. Each Ω_m is a singular link complement. The components of the link are labelled by ∞, while segments of the singular locus of cone angle π (resp. $2\pi/3$) are labelled by 2 (resp. 3).

3.1. **The cases Ω_m, $m = 15, 23, 47, 71$.** Notice that the singular locus of these four orbifolds consists of two unknotted circles—one of cone angle π, the other of cone angle $2\pi/3$—linked as shown:



Now we simply use the fact that the n-fold cyclic cover of S^3 branched over an unknotted circle is homeomorphic to S^3. Let $\Omega'_m \to \Omega_m$ be the 3-fold cyclic cover branched over the circle of cone angle $2\pi/3$. Then the orbifold Ω'_m is a link complement with singular locus an unknotted circle Σ of cone angle π. Hence the 2-fold cover $\widetilde{\Omega}_m \to \Omega'_m$ branched over Σ gives the desired nonsingular link complement.

We have drawn these links for $m = 15, 23, 47$ in Figures 2–4. As for $m = 71$, the complexity of the resulting link is such that we leave this as an exercise for the reader!

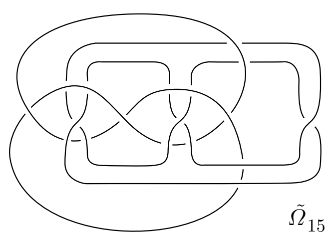


Figure 2.

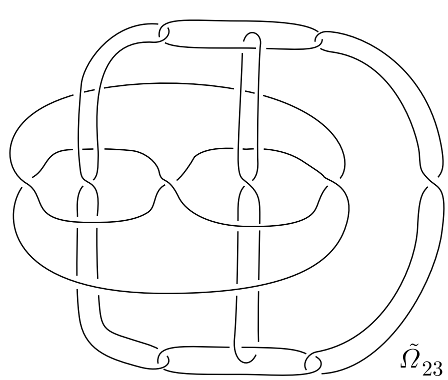
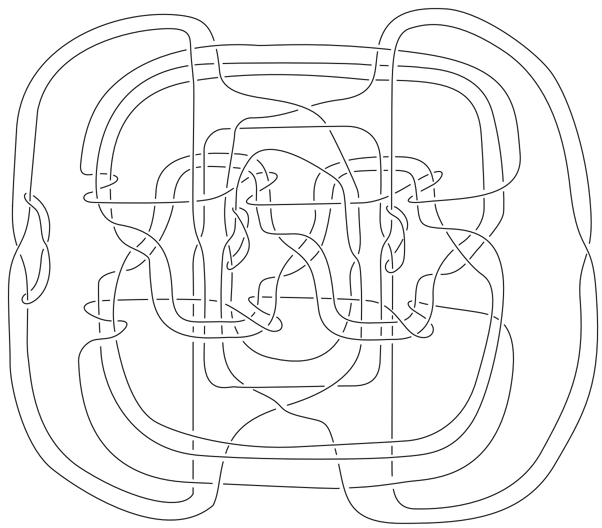


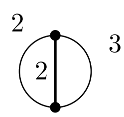
Figure 3.



$\tilde{\Omega}_{47}$

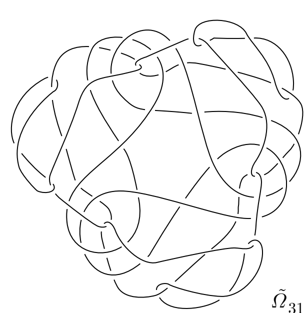
FIGURE 4.

3.2. **The case Ω_{31}.** The singular locus of Ω_{31} is:



The edges of cone angle π form an unknotted circle, Σ_2. Let $\Omega'_{31} \to \Omega_{31}$ be the 2-fold cyclic cover, branched over Σ_2. Thus the orbifold Ω'_{31}, is again a link complement, with singular locus an unknotted circle, Σ_3, of cone angle $2\pi/3$. Hence the 3-fold cyclic cover of Ω'_{31} branched over Σ_3 is a nonsingular link complement, drawn in Figure 5.

3.3. **The cases Ω_m, $m = 5, 6, 39$.** We construct the covers $\widetilde{\Omega}_m \to \Omega_m$ using, in addition to the above techniques, the following lemma.



$\tilde{\Omega}_{31}$

FIGURE 5.

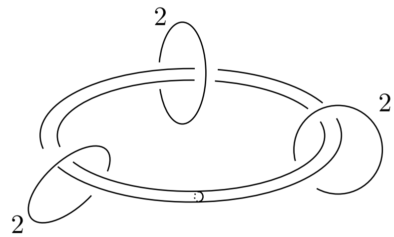
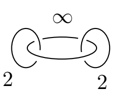
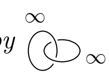


FIGURE 6a.

Lemma. (a) *The orbifold* 

is 2-fold branch covered by 

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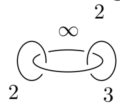
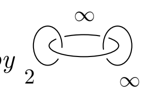
(b) *The orbifold* 

is 3-fold (irregular) branch covered by 

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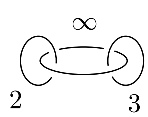
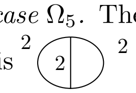
Proof. We prove (b). The proof of (a) is analogous. We begin by compactifying the orbifold (b) by removing a tubular neighborhood of its link. Now consider its 3-fold cyclic cover, branched over Σ_3 pictured in Figure 6a. The preimage of Σ_2 consists of three unknotted circles of cone angle π. We further identify two of these singular circles by cutting along their (punctured) bounding disks and then identifying D_1^+ with D_2^- (resp. D_1^- with D_2^+) by pushing everything through the tube, T (Figures 6b–6c). □

Notice that in these three cases, Ω_m contains the configuration of lemma (b):



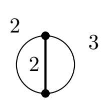
Let $\Omega'_m \to \Omega_m$ be the corresponding 3-fold irregular branched cover.

3.3.1. *The case* Ω_5. The construction of Ω'_5 is illustrated in Figure 7a. The singular locus of Ω'_5 is 

which we desingularize by taking cyclic covers as illustrated in Figure 7b. We obtain $\widetilde{\Omega}_5$ an 8-circle nonsingular link complement.

3.3.2. *The case* Ω_6. The 3-fold irregular cover $\Omega'_6 \to \Omega_6$ is drawn in Figure 8a. Now Ω'_6 contains the configuration of lemma (a); hence we obtain the 2-fold branched cover $\Omega''_6 \to \Omega'_6$ pictured in Figure 8b. The singular locus of Ω''_6 is one circle of cone angle π, so that its 2-fold branched cover yields the 12-circle link complement in Figure 8c.

3.3.3. *The case* Ω_{39}. The 3-fold irregular branched cover $\Omega'_{39} \to \Omega_{39}$ is drawn in Figure 9. The singular locus of Ω'_{39} is:



Thus (by applying verbatim the steps in §3.2) we know that Ω'_{39} is 6-fold covered by a nonsingular link complement. Given the complexity of this link, we do not draw it here.

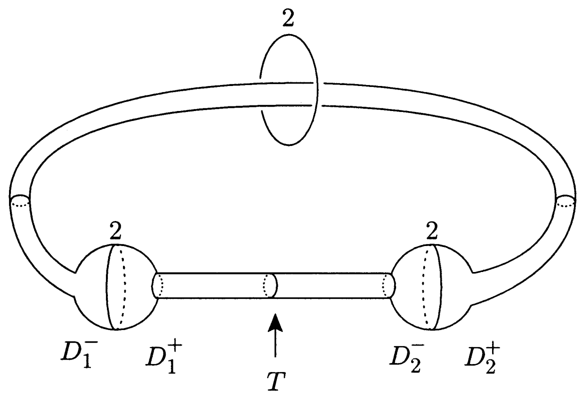


FIGURE 6b.

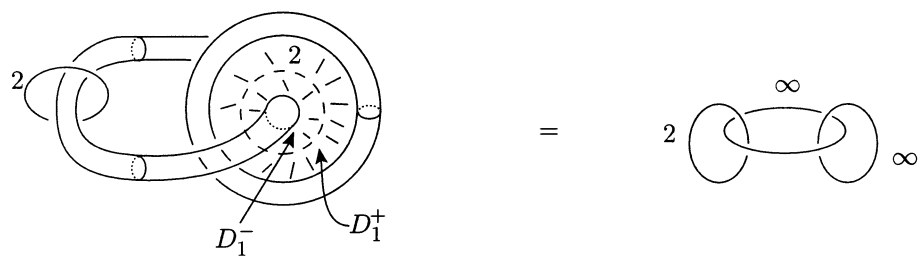


FIGURE 6c.

3.4. **The case Ω_{19}.** The orbifold Ω_{19}, owing to its complicated singular locus, was the hardest to deal with. We began by considering the quotient group

$$\mathrm{PSL}_2(\mathcal{O}_{19})/\left\langle\left\langle\begin{bmatrix}1 & \omega\\ 0 & 1\end{bmatrix}\right\rangle\right\rangle, \qquad \omega=\frac{-1+\sqrt{-19}}{2},$$

which the computer program GAP ([G]) found to be of order 60. This implies the following:

Proposition. *The congruence subgroup $\Gamma(I)\subset \mathrm{PSL}_2(\mathcal{O}_{19})$, $I=(\omega)$, is the fundamental group of a link complement in a homotopy 3-sphere.*

Proof. Recall that $\Gamma(I)$ is the kernel of $\mathrm{PSL}_2(\mathcal{O}_{19})\to \mathrm{PSL}_2(\mathcal{O}_{19}/I)$ (see [S] for details). Since $\mathcal{O}_{19}/I\cong \mathbb{Z}/5\mathbb{Z}$, it follows that $\Gamma(I)$ is a normal, torsion-free subgroup of index 60 in $\mathrm{PSL}_2(\mathcal{O}_{19})$ containing the matrix $\left[\begin{smallmatrix}1 & \omega\\ 0 & 1\end{smallmatrix}\right]$. Thus $\Gamma(I)=\langle\langle\left[\begin{smallmatrix}1 & \omega\\ 0 & 1\end{smallmatrix}\right]\rangle\rangle$ so that $\Gamma(I)$ is generated by $\mathrm{PSL}_2(\mathcal{O}_{19})$-conjugates of the parabolic element $\left[\begin{smallmatrix}1 & \omega\\ 0 & 1\end{smallmatrix}\right]$. Hence one can perform Dehn filling on the ends of $\mathbb{H}^3/\Gamma(I)$ so as to obtain a simply connected 3-manifold. ◻

We will show that $\mathbb{H}^3/\Gamma(I)$ is in fact a link complement in S^3 (which we do not draw) as well as give explicitly a 4-component link complement $\widetilde{\Omega}_{19}$ (Figure 10c), that is a 12-sheeted orbifold cover of Ω_{19}.

Let X denote Ω_{19} with its link circle put back (Dehn fill Ω_{19} with respect to the meridian $\left[\begin{smallmatrix}1 & \omega\\ 0 & 1\end{smallmatrix}\right]$). Note that X is topologically S^3. Both of the above results follow from the fact that X admits a 6-fold irregular orbifold cover, X', that is

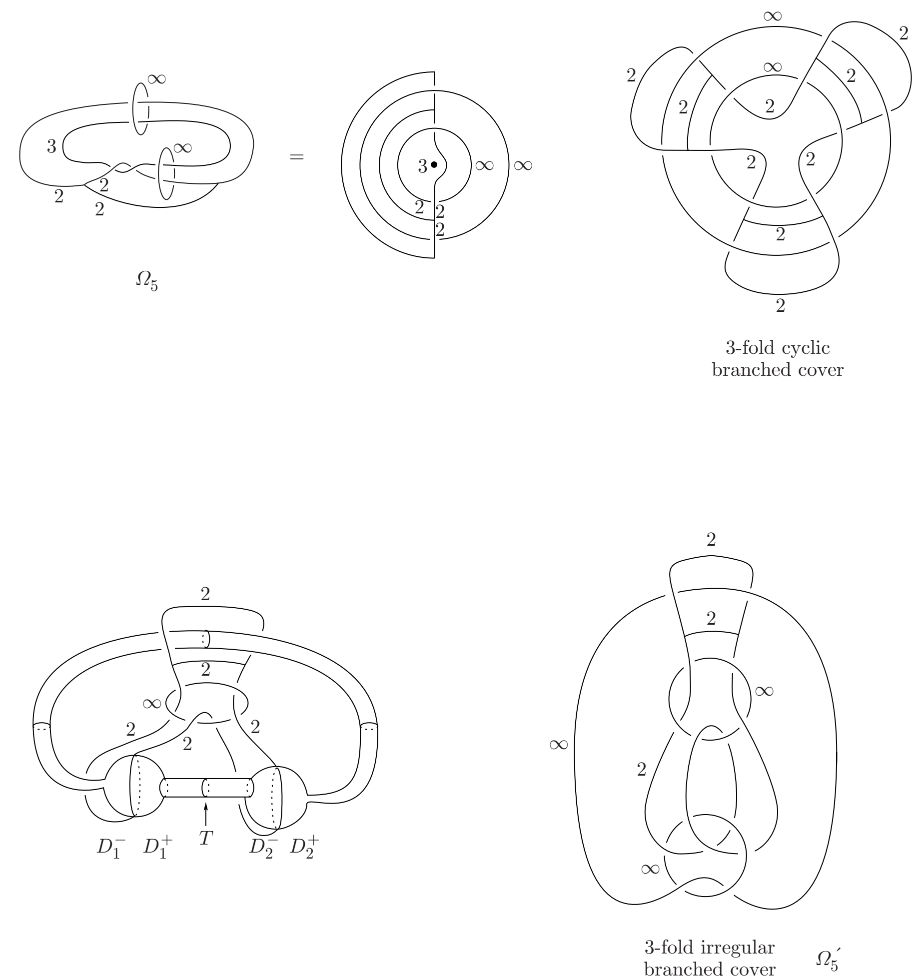


FIGURE 7a. Construction of Ω_5'

homeomorphic to S^3 with singular locus a figure eight knot of cone angle π. The construction of X' is carried out in §3.4.1 below (see Figures 10d–10j).

The existence of X' implies that X is 60-fold orbifold covered by a nonsingular S^3, for X' is 2-fold branch covered by the lens space $L(5,2)$ which is in turn 5-fold covered by S^3. Now, removing the link circle from X and its preimages from S^3 yields a 60-fold orbifold cover of Ω_{19} by a link complement which, we claim, corresponds to the group $\Gamma(I)$. Indeed, by construction the group Γ of the link complement is generated by conjugates of the matrix $\left[\begin{smallmatrix} 1 & \omega \\ 0 & 1 \end{smallmatrix}\right]$ so that $\Gamma \subset \Gamma(I)$ and thus $\Gamma = \Gamma(I)$ since both are of index 60 in $\mathrm{PSL}_2(\mathcal{O}_{19})$.

Actually constructing this link seemed rather difficult so we obtained the 4-component link complement $\widetilde{\Omega}_{19}$ (Figure 10c) as follows. Removing the link circle from X and its preimages from X' yields a 6-fold irregular orbifold cover, Ω_{19}', of Ω_{19} pictured in Figure 10a. Explicitly, one obtains Ω_{19}' by removing the link circle from X and performing the same cutting and pasting as in the construction of X'

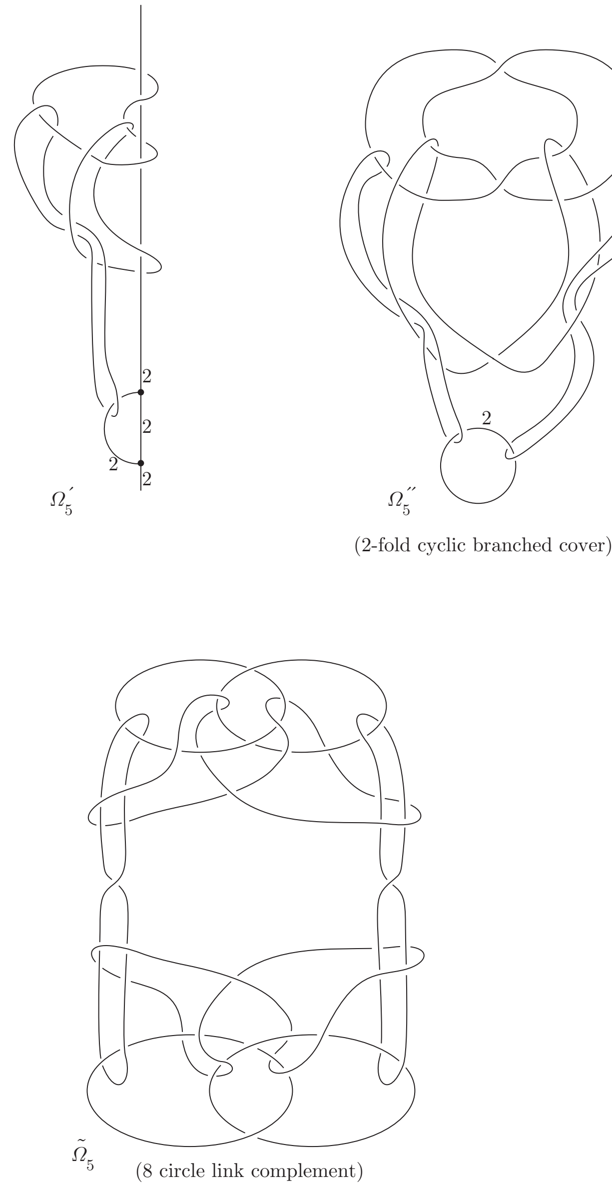


FIGURE 7b. Construction of $\widetilde{\Omega}_5$

given below. We omit the details. Now one easily obtains $\widetilde{\Omega}_{19}$ as illustrated in Figures 10b–10c: first twist in the top component of the link in Ω_{19}' so as to unknot the singular circle, then take the appropriate 2-fold branched cover.

3.4.1. We conclude by constructing a 6-fold irregular orbifold cover, X', of X homeomorphic to S^3 with singular locus a figure eight knot of cone angle π. This is done in steps 1–5 below (see Figures 10d–10j).

The orbifold X is drawn in Figure 10d. Remember that $\overset{n}{\text{---}}$ represents an edge of cone angle $2\pi/n$. We have added arrows to the edges of the singular locus to aid

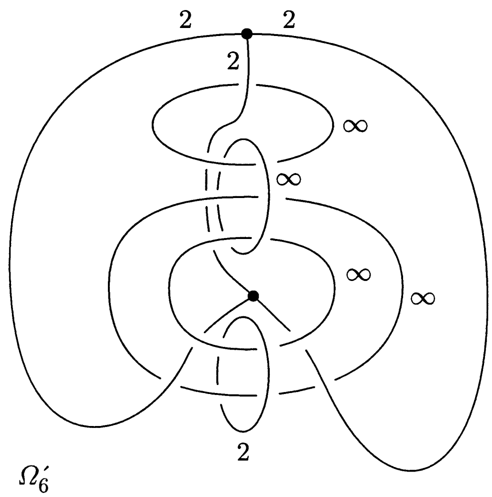


FIGURE 8a.

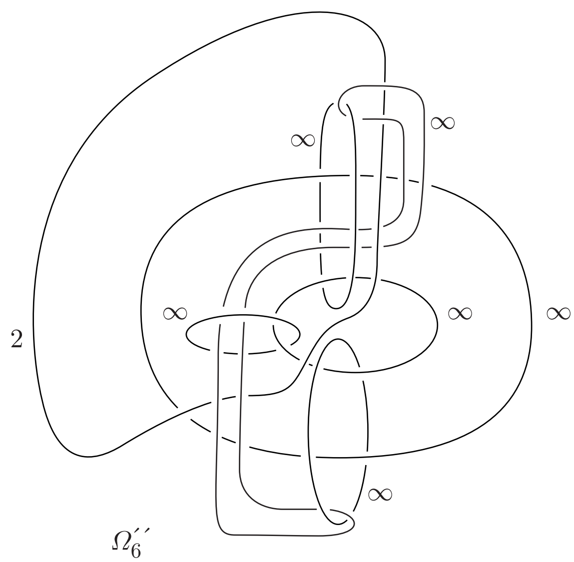


FIGURE 8b.

in following the construction. The orbifold X' is obtained by cutting and pasting together 6 copies of X along disks which are bounded by loops in the singular locus of X.

1. Redraw X as in Figure 10e. In Figure 10f we have split S^3 open along a disk D bounded by a loop in the singular locus of X. Note that the segment $\overset{3}{\twoheadrightarrow}$ has been split into two segments of cone angle $\pi/3$, one on each copy of D.

2. Glue together two copies of X along D as shown in Figure 10g to obtain the cone manifold X_2. Note that X_2 is not an orbifold cover of X, since the labels on its singular locus are not all integers.

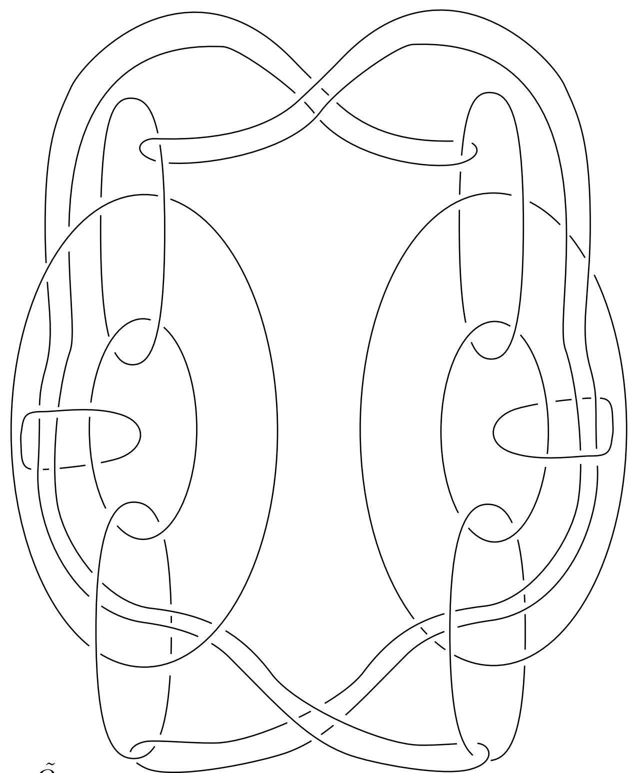


FIGURE 8c.

3. Glue a copy of X to X_2 from the "back" of X_2 (cut and paste along disk E) to get X_3 as pictured in Figure 10h. Note cancellation of singular loci 3 and 3/2 since $2\pi/3 + 2\pi/(3/2) = 2\pi$.

4. Now glue a copy of X to X_2 from the "front" (cut and paste along disk F) to get X_3' as pictured in Figure 10i.

5. Finally, gluing X_3 to X_3' (along disk G) gives the desired cover X' of X drawn in Figure 10j.

4. THE ORBIFOLDS Ω_m

The orbifold $\Omega_m = \mathbb{H}^3/\operatorname{PSL}_2(\mathcal{O}_m)$ is topologically a noncompact 3-manifold with cusps of the form $T^2 \times [0, \infty)$ for $m \neq 1, 3$ (resp. $S^2 \times [0, \infty)$ for $m = 1, 3$). In all cases, the number of cusps of Ω_m is equal to the class number of $\mathcal{O}_m$ (see [S]).

It is precisely for the 14 values of m in the list $\mathcal{L}$ that Ω_m embeds in S^3. The orbifolds for $m = 5, 6, 15, 19, 23, 31, 39, 47, 71$ are drawn in Figure 1.

We obtained these embeddings as follows.

4.1. **The cases** $m = 5, 6, 15, 19, 23, 31$**.** Let $\Gamma_m = \operatorname{PSL}_2(\mathcal{O}_m)$ and D_m be a Ford fundamental polyhedron for the action of Γ_m on $\mathbb{H}^3$ (see [Sw] for details of this construction. The domains D_m, $m \leq 19$, are also explicitly given). R. Riley has written a computer program (described in [R2]) that computes the Ford domains

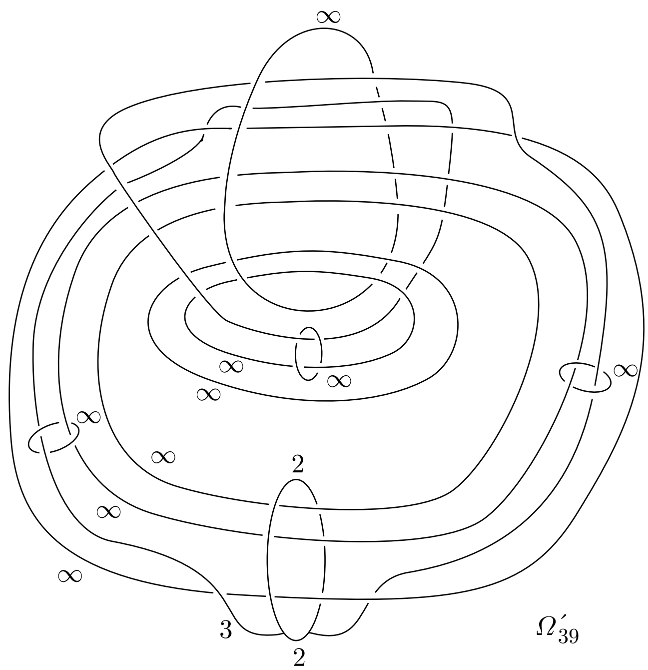


FIGURE 9.

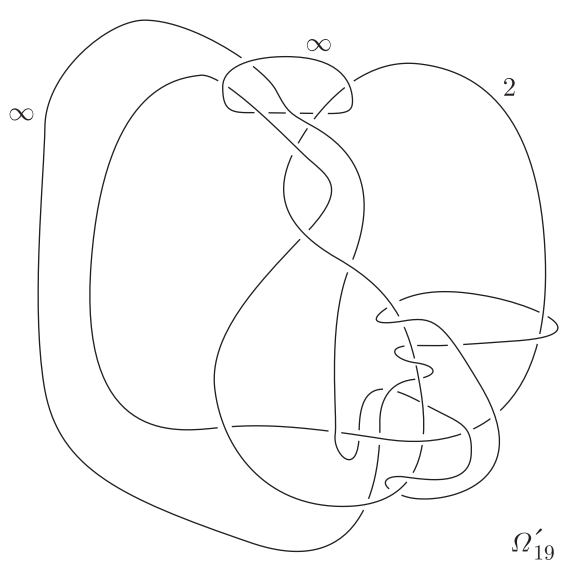


FIGURE 10a.

and face pairings for $\mathrm{PSL}_2(\mathcal{O}_m)$ and $\mathrm{PGL}_2(\mathcal{O}_m)$. Copies of the program/domains are available from him upon request.

Let D_m^* be the compactification of D_m obtained by removing a horoball neighborhood about each of the cusps.

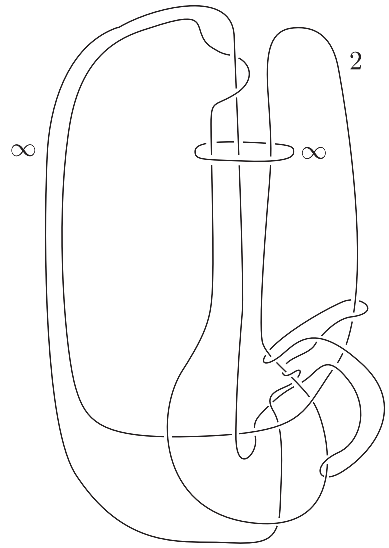


Figure 10b.

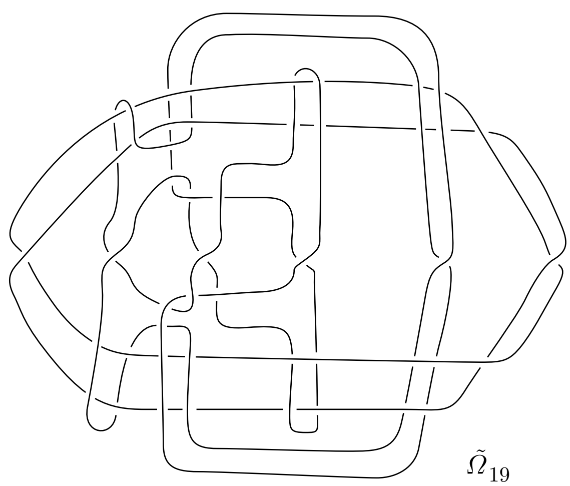


Figure 10c.

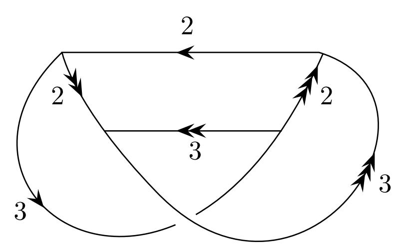


Figure 10d.

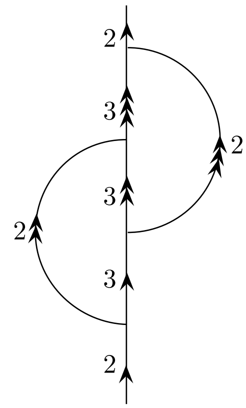


FIGURE 10e.

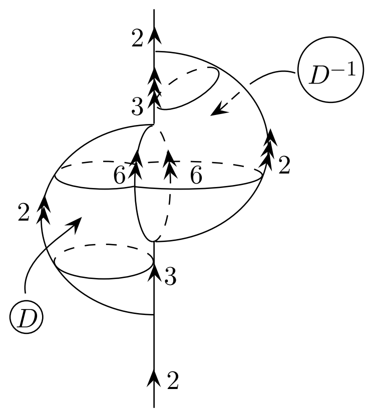


FIGURE 10f.

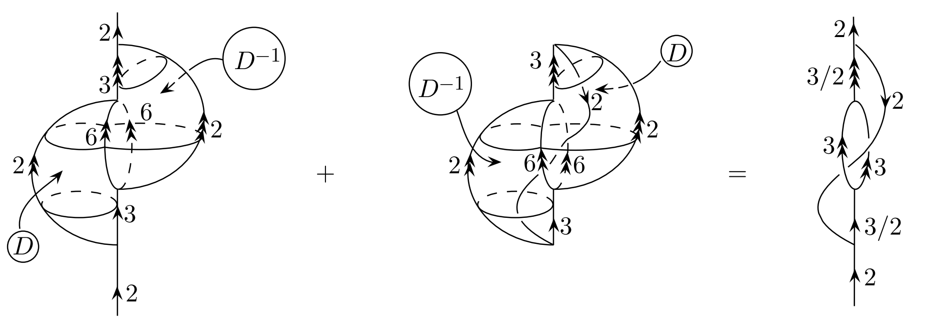


FIGURE 10g. Construction of X_2

For our purposes, the key facts concerning D_m^* $(m \neq 1, 3)$ are:

i) D_m^*/Γ_m is topologically a compact 3-manifold with interior homeomorphic to $\Omega_m = D_m/\Gamma_m$ and boundary a disjoint union of tori.

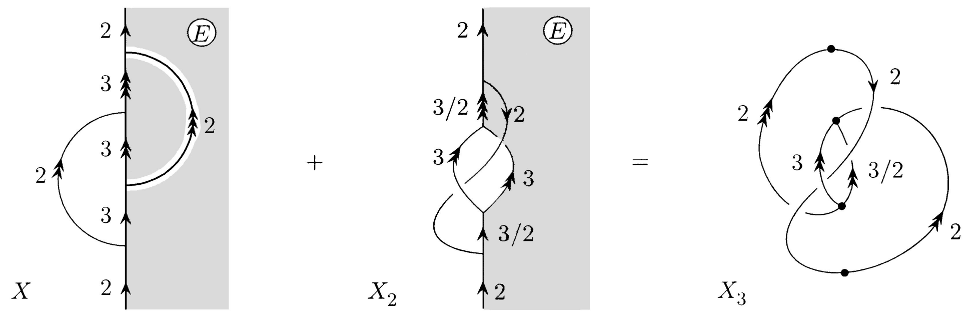


FIGURE 10h. Construction of X_3

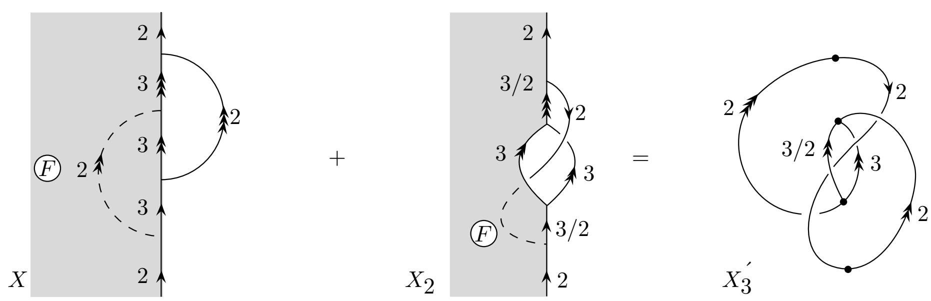


FIGURE 10i. Construction of X_3'

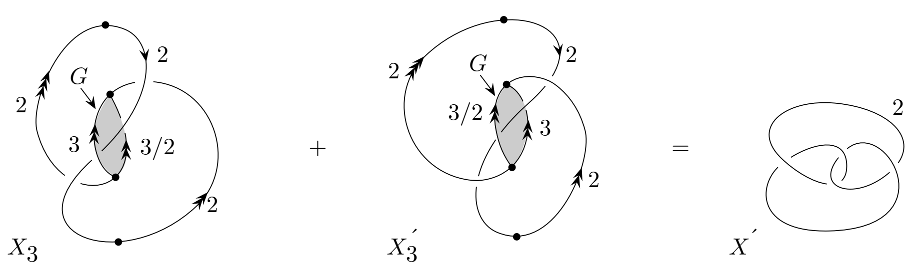


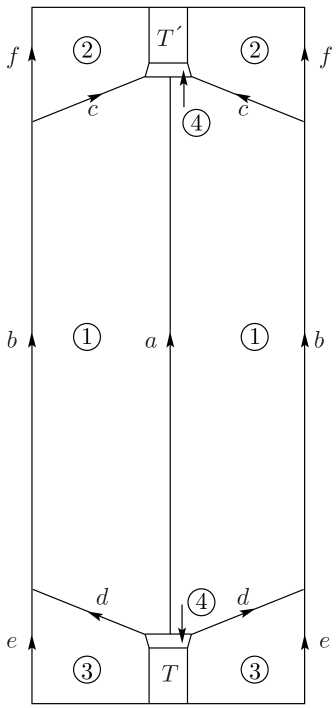
FIGURE 10j. Construction of X'

ii) Letting Γ_m^∞ denote the parabolic subgroup of Γ_m fixing $\{\infty\}$, then D_m^*/Γ_m^∞ is homeomorphic to $T^2 \times [0,1]$ with $T^2 \times \{1\}$ the compactification of the cusp at $\{\infty\}$ and $T^2 \times \{0\}$ containing the singular locus of Ω_m as well as the remaining unpaired faces of D_m^*.

Thus the first step in obtaining Ω_m is to embed $D_m^*/\Gamma_m^\infty \cong T^2 \times [0,1]$ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ so that:

1) $T^2 \times \{1\}$ is the boundary of an open tubular neighborhood of the circle, C, formed by the z-axis of $\mathbb{R}^3$ and the point $\{\infty\}$.

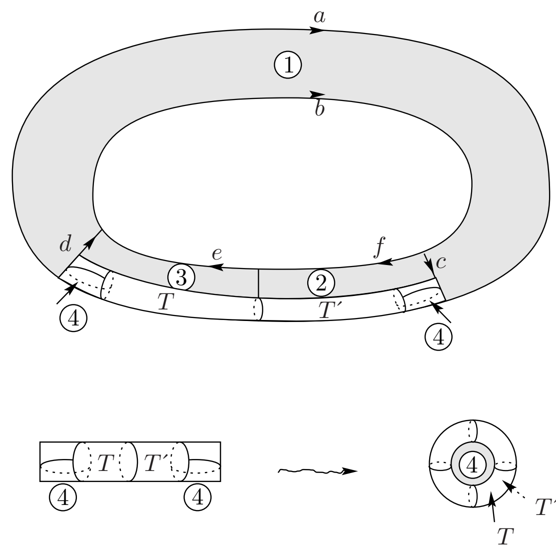
2) The loop on $T^2 \times \{1\}$ corresponding to the element $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ (resp. the element $\left[\begin{smallmatrix} 1 & \omega \\ 0 & 1 \end{smallmatrix}\right]$) of Γ_m is parallel to C (resp. links C).



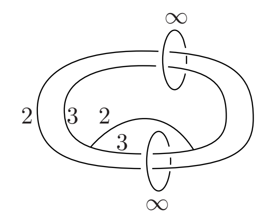
D_6^* (projection onto $\mathbb{C}$ with face pairings indicated)

Edges cone angle π: a, e, f

Edges cone angle $2\pi/3$: b, c, d



D_6^*/Γ_6^∞ with, in addition, face pairs ①, ②, ③ identified



Ω_6

FIGURE 11. Embedding Ω_6

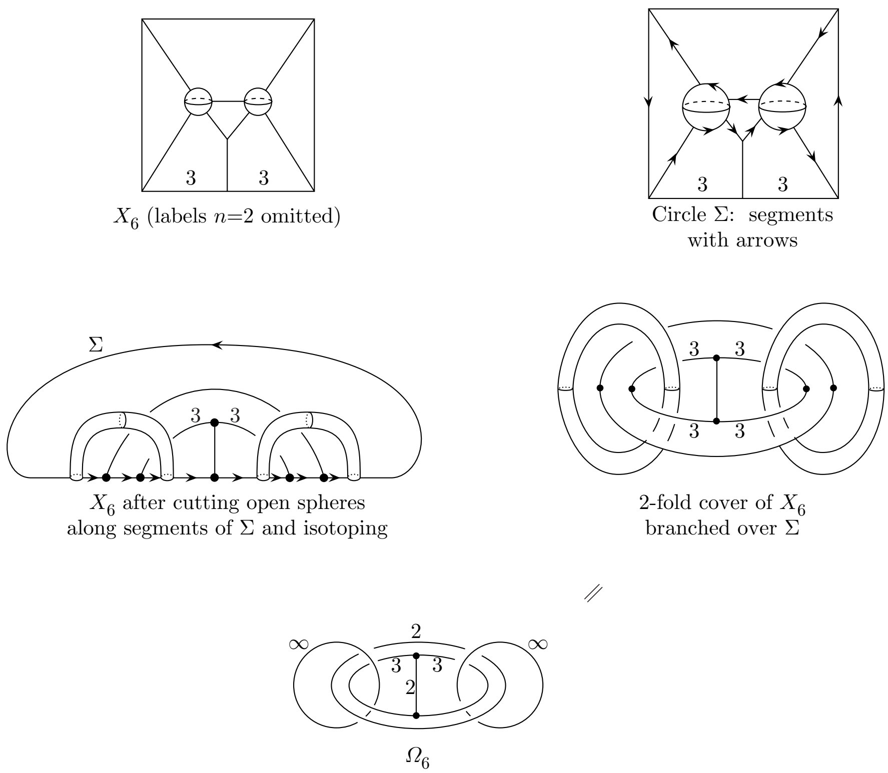


FIGURE 12. Obtaining Ω_6 from X_6

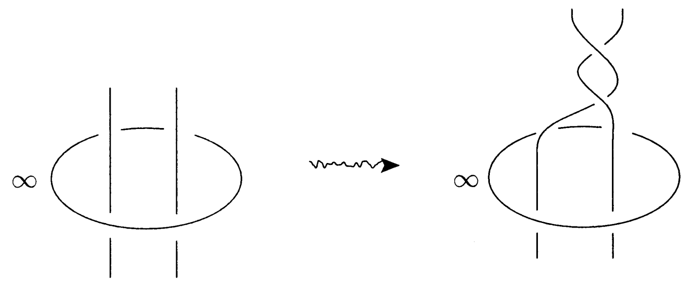
The next step is to identify the remaining faces on $T^2 \times \{0\}$ from the inside. As an example, the embedding of D_6^*/Γ_6 is illustrated in Figure 11.

4.2. **The cases** $m = 39, 47, 71$**.** After doing the above cases, we were motivated to study Allen Hatcher's work on $\mathrm{PGL}_2(\mathcal{O}_m)$ orbifolds, $X_m = \mathbb{H}^3/\,\mathrm{PGL}_2(\mathcal{O}_m)$ (see [H2]). Using Riley's $\mathrm{PGL}_2(\mathcal{O}_m)$ Ford domains, Hatcher draws the nineteen X_m that embed in S^3. These orbifolds have cusps of the form $S^2 \times [0, \infty)$, and (for $m \neq 1, 3$) Ω_m is a 2-fold branched cover of X_m branched in such a way so that the cusp spheres of X_m are covered by cusp tori in Ω_m. Indeed, $\mathrm{PGL}_2(\mathcal{O}_m)/\,\mathrm{PSL}_2(\mathcal{O}_m) \cong \mathbb{Z}/2\mathbb{Z}$, and one obtains $\mathrm{PGL}_2(\mathcal{O}_m)$ from $\mathrm{PSL}_2(\mathcal{O}_m)$ by adjoining the involution $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. It is this involution and its conjugates that act on the cusp tori of Ω_m transforming them into cusp spheres.

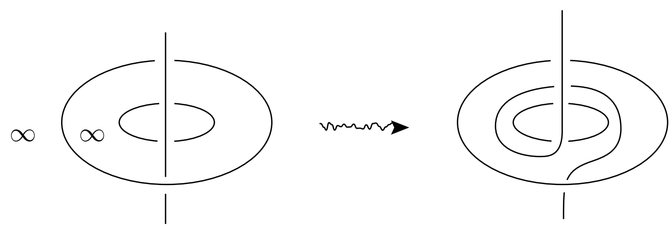
In each case ($m \neq 1, 3$) one can find an unknotted circle, Σ, in X_m consisting of edges of cone angle π and two disjoint arcs on each cusp sphere such that Σ contains all singular edges with one or both vertices on cusp spheres. One obtains Ω_m as the 2-fold cover of X_m branched over Σ. In addition to obtaining Ω_m for $m = 39, 47, 71$ in this manner, we also constructed Ω_m for $m = 5, 6, 15, 19, 23, 31$ and found that they agreed with the Ω_m constructed in §4.1. We illustrate this method in Figure 12 by obtaining Ω_6 from X_6. The reader can check that the orbifolds Ω_6 in Figures 11 and 12 are equivalent.

4.3. **Modification of embeddings: Twist homeomorphisms.** In certain cases, we have modified our embeddings of Ω_m in S^3 by:

1) Twisting about unknotted link components



2) Twisting about parallel link components



These twist homeomorphisms allow us to unknot singular circles as well as to link and unlink different components of a singular locus (see [Ro], Chapter 9, for a discussion of twist homeomorphisms).

The orbifolds Ω_m pictured in Figure 1 are embedded in a way that is most convenient for constructing the covers $\widetilde{\Omega}_m$, and hence may differ from the embeddings appearing elsewhere.

5. Volume of $\widetilde{\Omega}_m$

A nice property of the Bianchi orbifolds is the existence of a number theoretical formula, due to Humbert, for their volume (see [T], Chapter 7):

$$\mathrm{vol}(\Omega_m) = \frac{(D)^{3/2}\zeta_{\mathbb{Q}\sqrt{-m}}(2)}{24\zeta_{\mathbb{Q}}(2)}, \qquad D = \begin{cases} 4m, & m \equiv 1,2 \pmod 4, \\ m, & m \equiv 3 \pmod 4. \end{cases}$$

Thus the volume of Ω_m and hence that of $\widetilde{\Omega}_m$ can be computed to any degree of accuracy desired using the number theory program Pari ([P]).

As a check of our results, we used SnapPea ([W]) to analyze $\widetilde{\Omega}_m$, Ω'_{39} and Ω_{71} in Figures 1–10 and found that their volumes agreed with those obtained from

Humbert's formula. We give these volumes below:

$$\mathrm{vol}(\widetilde{\Omega}_5) = 50.4476311137\ldots$$
$$\mathrm{vol}(\widetilde{\Omega}_6) = 62.1860747738\ldots$$
$$\mathrm{vol}(\widetilde{\Omega}_{15}) = 18.8316833668\ldots$$
$$\mathrm{vol}(\widetilde{\Omega}_{19}) = 31.8377775733\ldots$$
$$\mathrm{vol}(\widetilde{\Omega}_{23}) = 38.6951532243\ldots$$
$$\mathrm{vol}(\widetilde{\Omega}_{31}) = 54.0663025806\ldots$$
$$\mathrm{vol}(\Omega''_{39}) = 82.787851850\ldots$$

(Ω''_{39} is the 2-fold branched cover of Ω'_{39}. Its singular locus is an unknotted circle of cone angle $2\pi/3$)

$$\mathrm{vol}(\widetilde{\Omega}_{47}) = 116.606541915\ldots$$
$$\mathrm{vol}(\Omega_{71}) = 37.533306130\ldots$$

Acknowledgments

I thank F. Bonahon for helpful correspondence which led me to consider the orbifolds Ω_m. I also thank A. Hatcher for providing me with his $\mathrm{PGL}_2(\mathcal{O}_m)$ orbifold drawings which greatly facilitated my treatment of the cases $m = 39, 47, 71$.

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